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A Bijection for Eulerian-equivalence Classes of Totally Cyclic Orientations

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Abstract. Gioan showed that the number of cycle reversing classes of totally cyclic orientations of a given graph can be calculated as an evaluation of the corresponding Tutte polynomial. We note that the concept of cycle reversing classes of orientations coincides with that of Eulerian-equivalence classes considered by Chen and Stanley, and Kochol. Based on this coincidence, we give a bijective proof of Gioan's result. Precisely, the main result of the paper is an algorithmic bijection between the set of Eulerian-equivalence classes of totally cyclic orientations and the set of spanning trees without internally active edges.

Key words. Tutte polynomials, reduced orientations, totally cyclic orientations, cycle reversing classes, Eulerian-equivalence classes, internal activity, external activity.

AMS classification. 05A99, 05C20.

1. Introduction

To generalize the chromatic polynomial of a graph, Tutte [17] introduced the dichromatic polynomial in two dual variables which we know as the Tutte polynomial. It has been extended to general matroids, see Crapo [6]. Many interesting invariants of graphs and matroids can be computed directly from these polynomials. For a wealth of information on the properties and applications of Tutte polynomials, see Brylawski and Oxley [3]. It is worth mentioning that the Tutte polynomials play an important role in statistical mechanics, where the partition functions are just simple variants of these polynomials; the Jones polynomials and the Kauffman polynomials in knot theory are also closely related to them. We refer the reader to Bollobás [2] and references therein for more details of these connections.

This paper is exclusively concerned with the Tutte polynomials of graphs, and the graphs considered here are assumed to be always connected. Given a connected graph G, the Tutte polynomial $T_G(x, y)$ of G is originally defined as

$$T_G(x, y) = \sum_{i,j} t_{ij} x^i y^j, \tag{1}$$

where t_{ij} is the number of (i, j)-trees, each of which is a spanning tree of G and has precisely i internally active edges and precisely j externally active edges, see [17]. Hence, $T_G(1, 0)$ equals the number of spanning trees without externally active edges, $T_G(0, 1)$ equals the number of spanning trees without internally active edges, and $T_G(1, 1)$ equals the number of all spanning trees.

Much work has been done on enumerations of special orientations of graphs by evaluating the Tutte polynomials. A well-known result of Stanley [15] states that the number of acyclic orientations of G is equal to $T_G(2, 0)$. Note that a general result was already given by Winder [18] in order to count the number of regions of a hyperplane arrangement. Greene and Zaslavsky [11] proved that the number of acyclic orientations with a unique source at a given vertex is the special value of the Tutte polynomial at $(1, 0)$. It was also known that $T_G(0, 2)$ enumerates the number of totally cyclic orientations of G, see [2, p.372], and $T_G(2, 1)$ counts the indegree sequences of orientations of G, see Stanley [16].

Recently, Gioan [9] introduced some equivalence classes of orientations defined by reversing directed cycles, directed cocycles, or both. With a different formulation, the cycle reversing classes are used by Chen and Stanley [5] to study the flow polynomials, where they are called Eulerian-equivalence classes. On the other hand, the cocycle reversing classes are used by Chen [4] to study the tension polynomials, where they are called cut-equivalence classes. In the next section we will show the equivalence between the two characterizations of cycle reversing classes, as well as the equivalence between the two characterizations of cocycle reversing classes. Note that Kochol has already considered the Eulerian-equivalence relation for totally cyclic orientations in [13] and the cut-equivalence relation for acyclic orientations in [12].

It was shown by Gioan [9] that the number of cocycle reversing classes of acyclic orientations is equal to $T_G(1, 0)$, see also [4], and the number of cycle reversing classes of totally cyclic orientations is equal to $T_G(0, 1)$, see also [5]. Using the convolution formula of the Tutte polynomials, which was bijectively proved by Etienne and Las Vergnas [7] and was explicitly stated by Kook, Reiner and Stanton [14], Gioan proved that the value $T_G(2, 1)$ enumerates the number of cycle reversing classes of orientations, $T_G(1, 2)$ enumerates the number of cocycle reversing classes of orientations, and $T_G(1, 1)$ enumerates the number of cycle-cocycle reversing classes of orientations.

The main objective of this paper is to give a bijective proof of the result that $T_G(0, 1)$ counts the Eulerian-equivalence classes of totally cyclic orientations of G based on the reformulation of the equivalence relation as described in [5, 13]. For the bijective proof, we will define a bijection between spanning trees without internally active edges and Eulerian-equivalence classes of totally cyclic orientations. We introduce the notion of reduced orientations, which can be taken as the representative elements of Eulerian-equivalence classes. Our bijection is actually a bijective map from the set of reduced orientations of totally cyclic orientations to the set of spanning trees without internally active edges, which is dual to the bijection of Gebhard and Sagan [8] between the set of acyclic orientations with a unique sink at a given vertex and the set of spanning trees without externally active edges. Since there exists one and only one acyclic orientation with the unique given sink in every cocycle reversing classes of acyclic orientations, we see that the bijection of Gebhard and Sagan leads to a bijection between cocycle reversing classes of acyclic orientations and spanning trees without externally active edges.

Gioan and Las Vergnas [10] also gave a bijection between the set of acyclic orientations with a unique sink at a given vertex and the set of spanning trees without externally active edges, and this bijection was produced from their general activity-preserving correspondence between spanning trees and orientations with respect to certain order of the edge set. Notably, Gioan and Las Vergnas [10] established a bijection between spanning trees with external activity zero and activity classes of acyclic orientations, and dually a bijection between spanning trees with internal activity zero and activity classes of totally cyclic orientations. Note that both the present bijection in this paper and the bijection of Gebhard and Sagan are the variants of an algorithmic bijection of Blass and Sagan in [1]. We should point out that these bijections are not activity-preserving, different from the bijections of Gioan and Las Vergnas in [10].

2. Notation and Terminology

Much of the notation will follow that of [2]. Let $G = (V, E)$ be a connected graph with vertex set V and edge set E, in which multiple edges and loops are allowed. We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. By a *cycle* of G we mean a 2-regular connected subgraph in G. By a *cut* of G we mean a set of edges between S and T for some partition $V = S \cup T$ of the vertex set, denoted by $[S, T]$. An edge subset of G is called a *cocycle* if it can be written as a cut. A minimal cut is called a *bond*. A connected graph is called a *tree* if it contains no cycles. A tree $F = (V', E')$ is called a *spanning tree* of G if $V' = V$ and $E' \subset E$.

Let us now review the definitions of internal activity and external activity of spanning trees. Suppose that the edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$ is linearly ordered, where $e_i < e_j$ for $i < j$. Fix a spanning tree F of G. For each edge e_i in F, we call $U_F(e_i) = \{e_j \in E(G) : (F - e_i) + e_j \text{ is a spanning tree}\}$ the cut defined by e_i. If e_i is the smallest edge of the cut it defines, we call e_i an *internally active edge* of F. Similarly, for each edge e_j not in F, we call $Z_F(e_j) = \{e_i \in E(G) : e_i \text{ is an edge on the unique cycle of } F + e_j\}$ the cycle defined by e_j. If e_j is the smallest edge of the cycle it defines, we call e_j an *externally active edge*. We say that a spanning tree has internal activity i and external activity j if there are precisely i internally active edges and precisely j externally active edges, denoted by an (i, j)-tree.

Given $e \in E$, let $G - e = (V, E\backslash\{e\})$. Thus $G - e$ is obtained from G by deleting the edge e. Let G/e be the multigraph obtained from G by contracting the edge e. It is well known that the Tutte polynomials defined by (1) satisfy the following recurrence relation

$$T_G(x, y) = \begin{cases} xT_{G/e}(x, y), & \text{if } e \text{ is a bridge,} \\ yT_{G-e}(x, y), & \text{if } e \text{ is a loop,} \\ T_{G-e}(x, y) + T_{G/e}(x, y), & \text{if } e \text{ is neither a bridge nor a loop,} \end{cases}$$

where $T_G(x, y) = 1$ if G is a graph consisting of isolated vertices.

Note that, for a given graph $G = (V, E)$, each edge $e \in E(G)$ is incident with two vertices $u, v \in V(G)$, and it can be assigned a direction either from u to v or from v to u, but not both. In particular, a loop has two directions from a vertex to itself. If the edge e is directed from u to v, then it is denoted by $\overrightarrow{uv}$. An orientation ε of G gives each edge a direction, and the digraph obtained in this way is denoted by (G, ε). Let $\mathcal{O}(G)$ denote the set of all orientations of G. A partial orientation of G is a subset of edges of an orientation of G.

Given a digraph (G, ε), a directed cycle is a cycle in which all edges have a consistent direction with respect to ε. Let $\mathcal{AO}(G)$ denote the set of all orientations without directed cycles, namely, the set of acyclic orientations. Given a cut $[S, T]$ of G, we denote by $(S, T)_\varepsilon$ the set of all edges going from S to T relative to ε, and by $(T, S)_\varepsilon$ the set of all edges going from T to S. A bond $[S, T]$ is called directed relative to ε if $(S, T)_\varepsilon = \emptyset$ or $(T, S)_\varepsilon = \emptyset$. In Fig. 1, both $[\{v_1\}, \{v_2\}]$ and $[\{v_3\}, \{v_2\}]$ are directed bonds of (G, ε). A cut is called directed if it can be decomposed into a disjoint union of directed bonds. A directed cocycle is a set of edges forming a cut $[S, T]$ such that $(T, S)_\varepsilon = \emptyset$ or $(S, T)_\varepsilon = \emptyset$. Note that a directed cut need not to be a directed cocycle. For example, as shown in Fig. 1, in the digraph (G, ε) the directed cut $[\{v_1, v_3\}, \{v_2\}]$ is not a directed cocycle, but in the digraph (G', ε') the directed cut $[\{v_1', v_3'\}, \{v_2'\}]$ is a directed cocycle. Let $\mathcal{BO}(G)$ denote the set of all orientations without directed cuts, namely, the set of totally cyclic orientations.

We also need the concept of cycle flippable edges. Fix an orientation ε of G and an edge $e = (u, v)$. If there exists a directed cycle containing both u and v in the digraph obtained from (G, ε) by deleting the edge e, then we call e a *cycle flippable edge* relative to ε.

Given an orientation $\varepsilon \in \mathcal{O}(G)$ and a vertex $v \in V(G)$, let $E^+(v, \varepsilon)$ be the set of edges taking v as the head and $E^-(v, \varepsilon)$ the set of edges taking v as the tail. Then the indegree of v is defined as the cardinality of $E^-(v, \varepsilon)$, and the outdegree of v is defined as the cardinality of $E^+(v, \varepsilon)$. Now let us recall two different characterizations of cycle reversing classes of orientations. Gioan [9] defined two orientations ε_1 and ε_2 to be equivalent if ε_1 can be obtained from ε_2 by reversing some cycle C in the digraph (G, ε_1) and vice versa. The transitive closure of this relation defines *cycle reversing classes* of orientations. The notion of *Eulerian-equivalence classes* was considered in [5, 13], where two orientations ε_1 and ε_2 are defined to be Eulerian-equivalent, denoted by $\varepsilon_1 \sim_e \varepsilon_2$, if the spanning subgraph induced by the edge set $\{e \in E(G) \mid \varepsilon_1(e) \neq \varepsilon_2(e)\}$ is an oriented Eulerian graph with respect to ε_1 or ε_2, i.e., the indegree is equal to the outdegree at each vertex in the subgraph, see

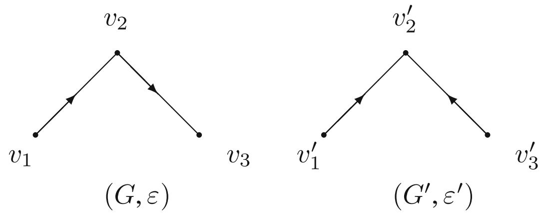

Fig. 1. Two digraphs.

Lemma 1 below. In fact, the notion of cycle reversing classes coincides with that of Eulerian-equivalence classes. We need the following variant of Euler's theorem.

Lemma 1. [2, p.16] *A digraph is Eulerian if and only if each vertex has the same outdegree and indegree.*

Proposition 1. (1) *The Eulerian-equivalence relation $\sim_e$ is indeed an equivalence relation on $\mathcal{O}(G)$, and it also induces an equivalence relation on $\mathcal{BO}(G)$.*
(2) *Two orientations ε_1 and ε_2 belong to the same cycle reversing class if and only if $\varepsilon_1 \sim_e \varepsilon_2$.*

Proof. (1) It is obvious that the relation $\sim_e$ is reflexive and symmetric. Suppose that $\varepsilon_1 \sim_e \varepsilon_2$ and $\varepsilon_2 \sim_e \varepsilon_3$. Then we proceed to prove that $\varepsilon_1 \sim_e \varepsilon_3$. Since reversing a directed cycle in a digraph doesn't change the indegree sequence, we have $|E^-(v,\varepsilon_1)| = |E^-(v,\varepsilon_2)|$ and $|E^-(v,\varepsilon_2)| = |E^-(v,\varepsilon_3)|$ for each $v \in V(G)$, and hence $|E^-(v,\varepsilon_1)| = |E^-(v,\varepsilon_3)|$. Let

$$\begin{aligned}
E_1 &= \{e \in E^-(v,\varepsilon_1) \mid \varepsilon_1(e) \neq \varepsilon_3(e)\},\\
E_2 &= \{e \in E^-(v,\varepsilon_1) \mid \varepsilon_1(e) = \varepsilon_3(e)\},\\
E_3 &= \{e \in E^+(v,\varepsilon_1) \mid \varepsilon_1(e) \neq \varepsilon_3(e)\}.
\end{aligned}$$

Note that

$$E^-(v,\varepsilon_1) = E_1 \uplus E_2, \quad E^-(v,\varepsilon_3) = E_2 \uplus E_3.$$

Thus, we have $|E_1| = |E_3|$, and from Lemma 1 it follows that the spanning subgraph induced by the edge set $\{e \in E(G) \mid \varepsilon_1(e) \neq \varepsilon_3(e)\}$ is an oriented Eulerian graph with respect to ε_1 or ε_3. This completes the proof of the desired transitivity of the relation $\sim_e$. Moreover, it is clear that if $\varepsilon_1 \sim_e \varepsilon_2$ and $\varepsilon_1 \in \mathcal{BO}(G)$, then ε_2 is also totally cyclic. Therefore, the relation $\sim_e$ is also an equivalence relation on $\mathcal{BO}(G)$.

(2) If $\varepsilon_1 \sim_e \varepsilon_2$, then by definition the spanning subgraph induced by the edge set $\{e \in E(G) \mid \varepsilon_1(e) \neq \varepsilon_2(e)\}$ is an oriented Eulerian graph with respect to ε_1, which can be written as a disjoint union of directed cycles $C_1, C_2, \ldots, C_r$. Therefore, ε_2 can be obtained from ε_1 by sequentially reversing $C_1, C_2, \ldots, C_r$. To prove the "only if" part, by transitivity of the Eulerian-equivalence relation, it suffices to show that $\varepsilon_1 \sim_e \varepsilon_2$ if ε_2 is obtained from ε_1 by reversing exactly one directed cycle (as is obvious from the definition of the Eulerian-equivalence relation). $\square$

Similarly to the definition of cycle reversing classes, Gioan [9] introduced the concept of *cocycle reversing classes* of orientations by reversing cocycles. In the same manner, we can show that the notion of cocycle reversing classes of orientations coincides with that of *cut-equivalence classes* of orientations considered in [4, 12]. Recall that two orientations ε_1 and ε_2 are defined to be *cut-equivalent*, denoted by $\varepsilon_1 \sim_c \varepsilon_2$, if the spanning subgraph induced by the edge set $\{e \in E(G) \mid \varepsilon_1(e) \neq \varepsilon_2(e)\}$ is a directed cut with respect to ε_1 or ε_2. Let C be a cycle of G with a given cyclic orientation ε_C. Given an orientation ε of G, let

$$[\varepsilon, \varepsilon_C](e) = \begin{cases} 1, & \text{if } e \in E(C) \text{ and } \varepsilon(e) = \varepsilon_C(e), \\ -1, & \text{if } e \in E(C) \text{ and } \varepsilon(e) \neq \varepsilon_C(e), \\ 0, & \text{if } e \notin E(C), \end{cases}$$

where $E(C)$ is the edge set of C. We need the following equivalent statement of the orthogonality of the cycle space and the cut space [2, p.53], see also [4, Proposition 6.2].

Lemma 2. *Given a non-empty subset $U \subseteq E(G)$ of a digraph (G, ε), then U is a directed cut if and only if for any directed cycle (C, ε_C),*

$$\sum_{e \in U \cap E(C)} [\varepsilon, \varepsilon_C](e) = 0.$$

Proposition 2. (1) *The cut-equivalence relation $\sim_c$ is indeed an equivalence relation on $\mathcal{O}(G)$, and it also induces an equivalence relation on $\mathcal{AO}(G)$.*

(2) *Two orientations ε_1 and ε_2 belong to the same cocycle reversing classes if and only if $\varepsilon_1 \sim_c \varepsilon_2$.*

Proof. (1) Obviously, the relation $\sim_c$ satisfies reflexivity and symmetry. Suppose that $\varepsilon_1 \sim_c \varepsilon_2$ and $\varepsilon_2 \sim_c \varepsilon_3$. Next we are to prove that $\varepsilon_1 \sim_c \varepsilon_3$. We may assume that $\varepsilon_1 \neq \varepsilon_3$. Let

$$E(\varepsilon_i \neq \varepsilon_j) = \{e \in E(G) \mid \varepsilon_i(e) \neq \varepsilon_j(e)\}, \quad \text{for } i, j = 1, 2, 3.$$

For any directed cycle (C, ε_C), by Lemma 2, we have

$$\begin{aligned}
&\sum_{e \in E(\varepsilon_1 \neq \varepsilon_3) \cap E(C)} [\varepsilon_1, \varepsilon_C](e) \\
&\quad = \sum_{e \in E(\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3) \cap E(C)} [\varepsilon_1, \varepsilon_C](e) + \sum_{e \in E(\varepsilon_1 \neq \varepsilon_2 = \varepsilon_3) \cap E(C)} [\varepsilon_1, \varepsilon_C](e) \\
&\quad = \left(\sum_{e \in E(\varepsilon_2 \neq \varepsilon_3) \cap E(C)} [\varepsilon_2, \varepsilon_C](e) - \sum_{e \in E(\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3) \cap E(C)} [\varepsilon_2, \varepsilon_C](e) \right) \\
&\qquad - \left(\sum_{e \in E(\varepsilon_1 \neq \varepsilon_2) \cap E(C)} [\varepsilon_2, \varepsilon_C](e) - \sum_{e \in E(\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3) \cap E(C)} [\varepsilon_2, \varepsilon_C](e) \right) \\
&\quad = \sum_{e \in E(\varepsilon_2 \neq \varepsilon_3) \cap E(C)} [\varepsilon_2, \varepsilon_C](e) - \sum_{e \in E(\varepsilon_1 \neq \varepsilon_2) \cap E(C)} [\varepsilon_2, \varepsilon_C](e) \\
&\quad = 0.
\end{aligned}$$

From Lemma 2 it follows that the spanning subgraph induced by the edge set $\{e \in E(G) \mid \varepsilon_1(e) \neq \varepsilon_3(e)\}$ is a directed cut with respect to ε_1 or ε_3. This completes the proof of the desired transitivity of the relation $\sim_c$. Moreover, it is clear that if $\varepsilon_1 \sim_c \varepsilon_2$ and $\varepsilon_1 \in \mathcal{AO}(G)$, then ε_2 is also acyclic. Therefore, the relation $\sim_c$ is also an equivalence relation on $\mathcal{AO}(G)$.

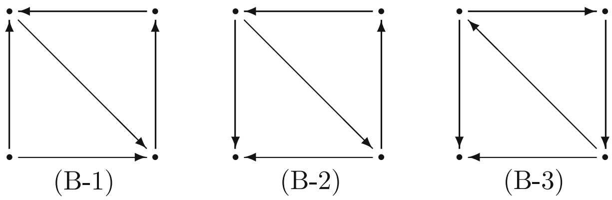

Fig. 2. Equivalence relations among three orientations.

(2) The proof is quite similar to that of (2) of Proposition 1; the details are omitted. □

It has been proven that both the Eulerian-equivalence relation and the cut-equivalence relation are equivalence relations, see [5,4], and the proofs are given here to make this paper self-contained. Gioan [9] also introduced the notion of *cycle-cocycle reversing classes* of orientations by reversing both cycles and cocycles. Note that for any given orientation ε of G the digraph (G, ε) is naturally decomposed into a cyclic part, which is the union of all directed cycles of G, and an acyclic part, which is the union of all directed cocycles of G. By the above two propositions, it is easy to verify that for $\mathcal{O}(G)$ the notion of cycle-cocycle reversing classes coincides with that of *Eulerian-cut equivalence classes* defined below. We call two orientations ε_1 and ε_2 Eulerian-cut-equivalent, denoted by $\varepsilon_1 \sim_{ec} \varepsilon_2$, if the spanning subgraph induced by the edge set $\{e \in E(G) \mid \varepsilon_1(e) \neq \varepsilon_2(e)\}$ is a disjoint union of an oriented Eulerian graph and a direct cut with respect to ε_1 or ε_2. According to the above propositions, the relation $\sim_{ec}$ is indeed an equivalence relation on $\mathcal{O}(G)$.

By definitions, the two orientations (B-1) and (B-2) in Fig. 2 are cut-equivalent, (B-2) and (B-3) are Eulerian-equivalent, while (B-1) and (B-3) are Eulerian-cut-equivalent.

3. Eulerian-equivalence Classes

The main goal of this section is to give a bijective proof of the following theorem on the enumeration of Eulerian-equivalence classes of totally cyclic orientations.

Theorem 1. [9] *For any graph G, let $\alpha(G)$ denote the number of Eulerian-equivalence classes of $\mathcal{BO}(G)$. Then*

$$\alpha(G) = T_G(0, 1). \tag{2}$$

Our proof is based on a bijection between Eulerian-equivalence classes of $\mathcal{BO}(G)$ and spanning trees without internally active edges, which is an algorithmic bijection similar to the modified Blass-Sagan algorithm [8].

Fix an orientation ε° of G (not necessarily totally cyclic or acyclic), which we will refer to as the *normal orientation*. Fix the total order imposed on the edge set $E(G)$ which is used to define the internal activity and the external activity. We say that an orientation ε is *reduced* if for each edge $e \in E(G)$ either $\varepsilon^{\circ}(e) = \varepsilon(e)$ or there exists no directed cycle containing e with all other edges on the cycle smaller than e in (G, ε).

Given a partial orientation ε of G, let G' be the graph obtained from G by contracting all undirected edges with respect to ε. Let ε' denote the orientation of G' inherited from ε. We say that ε is reduced if ε' is reduced with respect to the inherited normal orientation and the inherited order. Given any two partial orientations ε_1 and ε_2 of G such that the set of undirected edges of ε_1 coincides with that of ε_2, we say that they are Eulerian-equivalent if the two inherited orientations of the contraction G' are Eulerian-equivalent.

Lemma 3. *With the normal orientation ε° and the total order on edges fixed as above, there exists one and only one reduced orientation in each Eulerian-equivalence class of $\mathcal{BO}(G)$.*

Proof. We first show that for a given Eulerian-equivalence class there exists at least one reduced orientation. Start with an arbitrary totally cyclic orientation in the class, say ε_0. If ε_0 is reduced, then we are done. Otherwise, find the largest edge, say e_m, which doesn't satisfy the reduced property. It means that $\varepsilon^{\circ}(e_m) \neq \varepsilon_0(e_m)$ and there exists a directed cycle containing e_m such that all other edges on the cycle are smaller than e_m. By reversing the orientation of this cycle, we obtain another Eulerian-equivalent orientation ε_1 such that all edges larger than or equal to e_m satisfy the reduced property. Iterating the above process, we will get one orientation equivalent to ε_0, with all its edges satisfying the reduced property.

Now we show that the reduced orientation is unique in the Eulerian-equivalence class. Suppose there are two reduced equivalent orientations ε' and ε''. Consider the spanning subgraph H induced by the edge set $\{e \in E(G) \mid \varepsilon'(e) \neq \varepsilon''(e)\}$. If H is not empty, then the equivalence of ε' and ε'' implies that H is a disjoint union of directed cycles with respect to ε' or ε''. Taking a directed cycle of H with respect to ε', let e_m be the largest edge on this cycle. Without loss of generality, we may assume that $\varepsilon'(e_m) = \varepsilon^{\circ}(e_m)$. Therefore, $\varepsilon''(e_m) \neq \varepsilon^{\circ}(e_m)$, contradicting the assumption that ε'' is reduced. Thus the proof is complete. □

As shown above, for each Eulerian-equivalence class we can obtain the reduced orientation from any orientation ε in the class with the iterated process. For convenience we call it the *normalization* of ε.

In the following we will construct an algorithm which maps each reduced totally cyclic orientation to a spanning tree without internally active edges. Due to the above lemma, we obtain the desired bijection. As before, we impose a total order on the edge set $\{e_1, e_2, \ldots, e_{|E|}\}$ such that $e_i < e_j$ iff $i < j$. Then each directed edge will be sequentially examined and will be either deleted or unoriented using the following algorithm:

(S1) Input a graph (G, ε), where ε is a partial orientation of G.

(S2) Let (G', ε') be the contraction of (G, ε) with all undirected edges contracted. Reset ε' to be the reduced representation in the corresponding equivalence class.

(S3) Consider the largest edge e of G'. If e is a loop or a cycle flippable edge with respect to ε', then we delete e from (G', ε'). Otherwise, we unorient e in G'. Reset G to be the graph recovered from G' by uncontracting all undirected edges. Reset ε to be the orientation of G obtained from ε', i.e., for every directed edge e' we have $\varepsilon'(e') = \varepsilon(e')$. If G contains at least one directed edge with respect to ε, then go to Step (S2). Otherwise, go to Step (S4).

(S4) Output the graph G.

For an example of how the above algorithm works, see Fig. 3, where the symbol I denotes the unorientation, the symbol II denotes the deletion, and the symbol III denotes the normalization.

To show that this algorithm does produce a bijection, we shall first introduce a sequence of sets, $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q$ (where $q = |E(G)|$), such that $\mathcal{O}_0$ is the set of all reduced totally cyclic orientations of G, and $\mathcal{O}_q$ is the set of all spanning trees of G without internally active edges. We will show that the k-th step of the algorithm

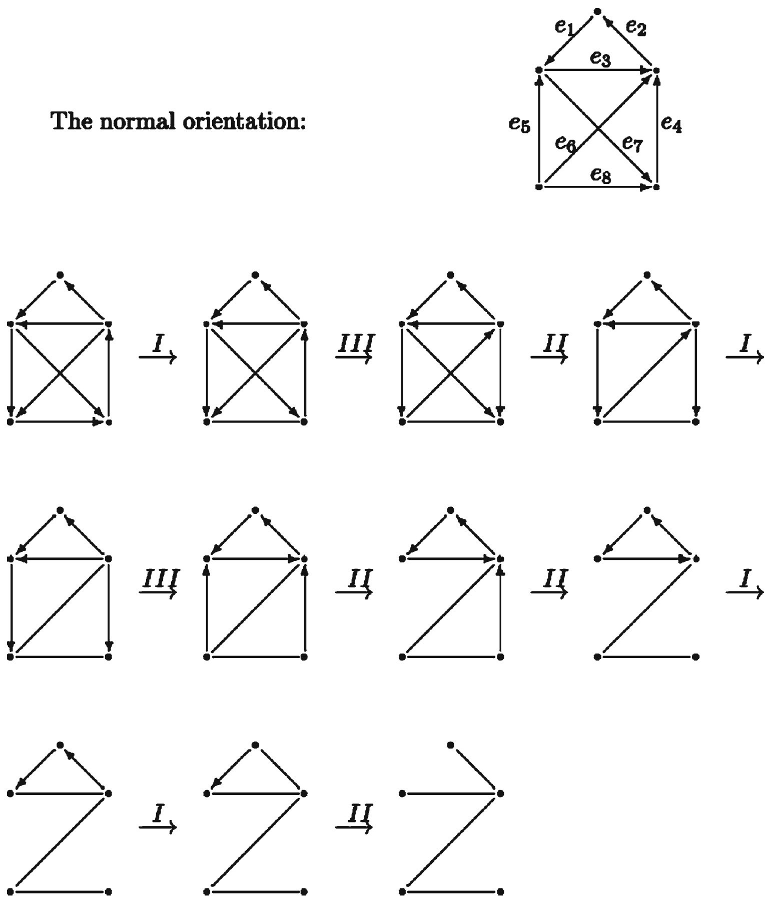

Fig. 3. An example of the algorithm.

gives a bijection, $f_k : \mathcal{O}_{k-1} \to \mathcal{O}_k$, where $\mathcal{O}_k$ is defined as the set of all partial orientations ε of spanning subgraphs of G satisfying the following conditions:

(a) Each of the first k largest edges of G is either present in ε (as an undirected edge) or absent from ε, but each of the remaining $q-k$ edges is present in ε in exactly one orientation, and there does not exist a cycle only consisting of undirected edges.
(b) The orientation obtained from ε by contracting all undirected edges is totally cyclic.
(c) The partial orientation ε is reduced.
(d) For each undirected edge e in the subgraph, if e is a bridge which separates the subgraph into two components C_1 and C_2, there exists at least one edge strictly smaller than e in the edge cut $E_G[C_1, C_2]$, the set of edges between the two components.

From the above conditions, we find that $\mathcal{O}_0$ is indeed the set of all reduced representations of the totally cyclic orientations of G, and $\mathcal{O}_q$ is indeed the set of all spanning trees without internally active edges.

Lemma 4. *f_k maps $\mathcal{O}_{k-1}$ into $\mathcal{O}_k$.*

Proof. It suffices to verify that properties (a)–(d) listed above are still satisfied after the algorithm is applied at the k-th stage.

(a′) If the k-th largest edge e is cycle flippable then the algorithm will delete it; otherwise, the algorithm will unorient it. Therefore, it will not create a new cycle consisting of only undirected edges.
(b′) Clearly, to unorient an edge or to delete a cycle flippable edge will not destroy the totally cyclic property.
(c′) This is ensured by Step (S2) of the algorithm.
(d′) Suppose that there exists some undirected edge e as a bridge in the subgraph such that e is the smallest edge in the edge cut $E_G[C_1, C_2]$. Then, in the process of the algorithm all edges of $E_G[C_1, C_2]$ except e must have been deleted at previous steps. Consider the second smallest edge e_0 of $E_G[C_1, C_2]$. Clearly, e_0 must not be cycle flippable when it was examined, and the algorithm will unorient it. This is a contradiction.

Thus we complete the proof. □

To prove that f_k is bijective, we first give the following two lemmas:

Lemma 5. *Given an orientation $\varepsilon \in \mathcal{O}_{k-1}$, let e be the largest directed edge of the underlying graph G. Let ε' be the orientation of $G-e$ inherited from ε. If ε is reduced and e is cycle flippable relative to ε, then ε' is also reduced. Moreover, $f_k(\varepsilon(G)) = \varepsilon'(G-e)$.*

Proof. Suppose that ε' is not reduced. There must exist one edge e' which is smaller than e and doesn't satisfy the reduced property in $G-e$. Clearly, e' also doesn't satisfy the reduced property for the orientation ε in G, which contradicts the hypothesis that ε is reduced. □

Lemma 6. *Given any two distinct reduced totally cyclic orientations ε_1 and ε_2 of G, suppose that the largest directed edge e is neither cycle flippable with respect to ε_1 nor with respect to ε_2. Let ε_1' (resp. ε_2') be the orientation of G/e inherited from ε_1 (resp. ε_2). Then ε_1' and ε_2' are not Eulerian-equivalent.*

Proof. Since $\varepsilon_1, \varepsilon_2$ are reduced and e is the largest edge in G, we must have $\varepsilon_1(e) = \varepsilon_2(e)$. Suppose that ε_1' and ε_2' are Eulerian-equivalent, then the edge set $\{e' \in E(G/e) \mid \varepsilon_1'(e') \neq \varepsilon_2'(e')\}$ can be written as a disjoint union $\cup_i C_i$, where each C_i is a directed cycle in G/e with respect to ε_1' or ε_2'. The set $\{e' \in E(G/e) \mid \varepsilon_1'(e') \neq \varepsilon_2'(e')\}$ can not be empty, otherwise we will have $\varepsilon_1 \sim_e \varepsilon_2$, contradicting the hypothesis that they are distinct reduced orientations. If for each i the edges in G corresponding to the edges of C_i also form a cycle, then we also have $\varepsilon_1 \sim_e \varepsilon_2$. Otherwise, suppose for some i the edges in G corresponding to the edges of C_i do not form a cycle, but together with the edge e they will form a cycle. If C_i and e form a directed cycle with respect to ε_1 (resp. ε_2), then e will be cycle flippable with respect to ε_2 (resp. ε_1) since ε_2 (resp. ε_1) is totally cyclic, again a contradiction. □

Theorem 2. *f_k is bijective.*

Proof. First we proceed to prove that f_k is one-to-one. Suppose ε_1 and ε_2 are two distinct elements of $\mathcal{O}_{k-1}$ which are both mapped to $\varepsilon \in \mathcal{O}_k$ by the algorithm. Since the algorithm only affects the k-th largest edge, we note that in both ε_1 and ε_2, the cases are the same for the first $k-1$ largest edges of G. We note that ε can not be obtained from ε_1 and ε_2 by deletion. Otherwise, ε_1 and ε_2 would be the same due to Lemma 5. Thus we only need to consider the case where ε was obtained from ε_1 and ε_2 by unorienting the k-th largest edge and applying the normalization. By Lemma 6, this is also impossible.

Then we prove that f_k maps $\mathcal{O}_{k-1}$ onto $\mathcal{O}_k$. For any $\varepsilon \in \mathcal{O}_k$ such that the k-th largest edge e of G is absent in the underlying spanning subgraph, we just add the edge e in the subgraph and normally orient it. Denote the orientation of this new digraph by ε'. Since ε is totally cyclic and the underlying graph is connected, ε' is still totally cyclic. Notice that e is the largest directed edge with respect to ε'. Therefore, ε' is also reduced and the edge e is cycle flippable. It means that $\varepsilon' \in \mathcal{O}_{k-1}$, and the k-th stage of the algorithm will map ε' to ε.

For any $\varepsilon \in \mathcal{O}_k$ such that the k-th largest edge $e = (u, v)$ of G is undirected in the underlying spanning subgraph, we construct an orientation $\varepsilon' \in \mathcal{O}_{k-1}$ as follows.

(1) Choose a direction of e such that the new orientation is totally cyclic. Note that such an orientation always exists.
(2) Normalize the new orientation. If the edge e is not cycle flippable, then return the orientation; otherwise, go to (3).
(3) Reorient the edge e oppositely, then go to (2).

Suppose that e is directed from u to v for the normal orientation. Note that for the orientation in Step (1) the outdegree of u is finite in the graph obtained by contracting all undirected edges, and each time when we reorient the edge e in Step

(3) the outdegree of u will be decreased by one. Therefore, the above procedure will terminate at some stage. Let ε' be the returned orientation. Clearly, e is not cycle flippable with respect to ε', and $\varepsilon' \in \mathcal{O}_{k-1}$. The k-th stage of the algorithm will map ε' to ε. This completes the proof. $\square$

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